Logic and reasoning will help you in many parts of your life. In talking with friends, listening to political speeches, and learning math, science, and even English literature and composition, you need to be able to understand and create convincing arguments.

Math understanding grows like a tree. It starts with definitions and postulates as its roots and then grows branches (theorems) that extend the tree higher and wider. How do you build or identify solid new branches? You do it through sound mathematical reasoning.

This unit is about mathematical reasoning and proof. With the tools of solid reasoning under your belt, you will be able to write and understand convincing arguments.

UNIT OBJECTIVES

► Distinguish between formal and informal methods of proof and explain why deductive reasoning is considered proof.

► Identify and define a conditional statement and form the inverse, converse, and contrapositive of the statement.

► Create truth tables for conditional statements and compound sentences.

► Reach conclusions from logical chains.

► Determine whether an argument is valid or invalid.

► Prove statements indirectly by use of a counterexample.

► Determine whether a statement is a definition and create definitions.

► Identify and use the algebraic and equivalence properties of equality.

► Write proofs in two-column and paragraph formats.

► Understand the difference between inductive and deductive reasoning.
Reasoning, Arguments, and Proof

When you heat tightly sealed food in the microwave oven, pressure builds and the lid pops off the container or the food explodes out of the wrapper. You quickly discover that if you continue to seal the food tightly when you heat it, you will have a yucky mess!

Reasoning in which you make general assumptions from specific observations is called induction. When you use deductive reasoning, you progress in the opposite direction, from the general to the specific. You collect observations and test your hypotheses with thorough information and statistics to confirm or disprove your original assumptions.

Induction—like the way you learn not to heat tightly sealed food in the microwave—is less exact and more exploratory. Deduction is controlled and tests hypotheses through previously established properties. Deduction is proof—induction is not.

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**KEYWORDS**

- argument
- conclusion
- deductive reasoning
- diagonals
- premise
- proof
- syllogism
- valid argument

**Reasoning**

Observations can be used to draw conclusions. You can use a table to keep track of observations and see if a pattern emerges. Look at the polygons below. As you look from left to right, the number of sides in each polygon increases by one.

The red lines are diagonals. **Diagonals** are segments that connect two vertices of a polygon and do not lie along any side of the polygon. For each polygon above, every diagonal that can be drawn from one vertex is shown. The triangle has no diagonals because any line connecting two vertices would lie along a side of the triangle. The numbers of sides and diagonals are listed in the table below.

<table>
<thead>
<tr>
<th>Number of Sides of Polygon (n)</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Diagonals</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>
If you study the table for a bit, you will see that there is a relationship between the number of sides of a polygon and the number of diagonals that can be drawn from one vertex of a polygon. So, from a simple observation, you can conclude that the number of diagonals that can be drawn from a single vertex is three less than the number of sides, or \( n - 3 \).

The conclusion about the number of diagonals was based on observation and studying of patterns. You can also derive conclusions on the basis of measurement and experimentation. For example, if you measure both diagonals of any rectangle, you will find that their lengths are always equal.

It is important to note that while observation, measurement, and experimentation can lead you to useful assumptions, those tools are not considered proof. Mathematicians rely on **deductive reasoning**, which uses previously proven or accepted properties to reach formal conclusions.

**Logic and Proof**

An **argument** is a set of statements, called **premises**, which are used to reach a **conclusion**. Both the premises and the conclusion are considered to be part of the argument.

A **syllogism** is a special kind of logical argument. It always contains two premises and a conclusion. Syllogisms have the following form.

- Premise: If \( a \), then \( b \).
- Premise: If \( b \), then \( c \).
- Conclusion: Therefore, if \( a \), then \( c \).

Example:

If Fido is hungry in the morning, then he barks.
If Fido barks, then Jenny wakes up.
Therefore, if Fido is hungry in the morning, then Jenny wakes up.

The validity of an argument is based on the structure of the argument. The syllogism above is a type of valid argument. In a **valid argument**, if the premises are all true, then the conclusion must also be true.

The following is an example of an invalid argument. The structure of the argument is faulty. Notice that both premises can be true and the conclusion can be false.

If you are on a baseball team, then you wear a red hat.
If you are a fireman, then you wear a red hat.
Therefore, if you are on a baseball team, then you are a fireman.
Proofs

Earlier in the book, proof was defined as a clear, logical structure of reasoning that begins from accepted ideas and proceeds through logic to reach a conclusion. In other words, a proof uses deductive reasoning. In a proof, only valid arguments are used, so the conclusions must be valid. Forms of valid arguments and different types of formal proofs will be shown throughout this unit.

Summary

- Observation, measurement, and experiments are useful, but they are not methods of proof. A formal proof uses deductive reasoning to reach a conclusion.
- An argument is a set of statements that are made up of a set of premises and a conclusion. The premises provide support for the conclusion.
- A syllogism is a special kind of logical argument that contains two premises and one conclusion.
- In a valid argument the premises cannot be true and the conclusion false at the same time.
Conditional Statements

A person who lives in Goodmath must be at least 17 years old to obtain a full driver’s license. Nathan, who lives in Goodmath, is over the age of 17. Although the two previous statements are true, can you conclude that Nathan has a driver’s license? No, there is not enough evidence. Nathan probably has a driver’s license, but you cannot be certain of it. Nathan’s sister, Julie, has a driver’s license and lives in Goodmath. Can you presume that she is over the age of 17? Yes, this deduction, based on previously known truths, does follow from the given information.

Be aware that although deductive reasoning seems to be an easy concept it’s possible for someone to be tricked into faulty reasoning. There is a definite difference between an untrue hypothesis and an argument with flawed logic.

KEYWORDS

- conclusion
- conditional statement
- contrapositive
- converse
- hypothesis
- inverse
- logical chain
- statement
- truth-functionally equivalent

Conditional Statements

A statement is a sentence that is either true or false. A conditional statement is a statement that has two parts. The first part begins with the word if and the second part begins with the word then. The hypothesis includes the words following if and the conclusion includes the words following then. In the conditional statement “If it is sunny, then I will mow the grass,” the words in blue are the hypothesis and the words in red are the conclusion.

Forms of Conditional Statements

By rearranging and negating the hypothesis and the conclusion, you can form the converse, inverse, and contrapositive of a conditional statement. The converse switches the hypothesis and the conclusion. The inverse negates, or takes the opposite of both the hypothesis and the conclusion. The contrapositive both switches and negates the hypothesis and the conclusion.
Conditional Statement  If it is sunny, then I will mow the grass.
Converse       If I mow the grass, then it is sunny.
Inverse        If it is not sunny, then I will not mow the grass.
Contrapositive If I do not mow the grass, then it is not sunny.

You can use letters and symbols to represent the forms of these statements. In logic, the letters \( p \) and \( q \) are usually used. When you use a variable in mathematics, you are using the variable to represent a number. In the same way, the letters \( p \) and \( q \) represent a group of words.

You can use the symbol \( \rightarrow \) to mean “implies” in a conditional statement. When you write \( p \rightarrow q \), you are saying that \( p \) implies \( q \), which is the same as if \( p \), then \( q \). The symbol \( \sim \) is read as “not.”

<table>
<thead>
<tr>
<th>Conditional Statement</th>
<th>If ( p ), then ( q )</th>
<th>( p \rightarrow q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Converse</td>
<td>If ( q ), then ( p )</td>
<td>( q \rightarrow p )</td>
</tr>
<tr>
<td>Inverse</td>
<td>If ( \sim p ), then ( \sim q )</td>
<td>( \sim p \rightarrow \sim q )</td>
</tr>
<tr>
<td>Contrapositive</td>
<td>If ( \sim q ), then ( \sim p )</td>
<td>( \sim q \rightarrow \sim p )</td>
</tr>
</tbody>
</table>

Both the hypothesis and the conclusion can be either true or false. This is called the truth value of each part. The truth values of those clauses determine whether the entire conditional is either true or false.

**Truth Tables**

Truth tables are an organized way to look at the possible truth values of an expression. The truth table for a conditional statement lists every possibility of truth values for the hypothesis, conclusion, and statement, where T means true and F means false.

**Conditional Statement** The truth table for a conditional statement is shown to the left. Notice that the only time \( p \rightarrow q \) is false is if the hypothesis is true and the conclusion is false. In the mowing example, this happens when it is sunny and I do not cut the grass. When both the hypothesis and the conclusion are true, the conditional statement is true. In other words, if it is sunny, then I cut the grass. If the hypothesis is false, the conditional statement is true, regardless of the conclusion. You don’t know what I will do if it isn’t sunny, so the conditional statement is true by default.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \rightarrow q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
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<tr>
<td>T</td>
<td>F</td>
<td>F</td>
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<td>T</td>
<td>T</td>
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<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

**Converse** When looking at the truth table for the converse, remember that \( q \) is now the hypothesis and \( p \) is the conclusion. So the converse is only false when \( q \) is true and \( p \) is false.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( q \rightarrow p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
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<tr>
<td>T</td>
<td>F</td>
<td>T</td>
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<td>F</td>
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<td>F</td>
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<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>
**Inverse** To make the truth table for the inverse, negate each clause first. Notice that the truth values of \( \sim p \) and \( \sim q \) are opposite those for \( p \) and \( q \). The inverse is only false when \( p \) is false and \( q \) is true.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( \sim p )</th>
<th>( \sim q )</th>
<th>( \sim p \rightarrow \sim q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
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<tr>
<td>T</td>
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<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
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</tbody>
</table>

**Contrapositive** To make the truth table for the contrapositive, negate \( p \) and \( q \) and then remember to switch the order so \( \sim q \) is the hypothesis. The conclusion is only false when \( p \) is true and \( q \) is false.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( \sim p )</th>
<th>( \sim q )</th>
<th>( \sim q \rightarrow \sim p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
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<td>T</td>
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</tbody>
</table>

Look at the the far right column in each table. The converse and the inverse columns match. Likewise, the original conditional statement and the contrapositive columns match. Those pairs of statements are **truth-functionally equivalent**—they are either both true or both false.

Most important is that the conditional statement and the contrapositive are equivalent. Because you can write any statement in “if-then” form, this will actually double the number of postulates and theorems you have. It also means that if you prove one of the statements, then you have also proven the other.

**Law of Contrapositives** If you prove the contrapositive of a conditional statement, then you also prove the conditional statement and vice versa.

**Euler Diagrams**

Euler diagrams are used to show how the parts of conditional statements are related. The inner circle represents the hypothesis, and the outer circle represents the conclusion. If a point lies in the inner circle, then it also lies in the outer circle. Or, said another way, “If \( p \), then \( q \).”
The Euler diagram below represents the conditional statement, which is “If it is sunny, then I will mow the grass.” If a point lies in the “it is sunny” circle, then it also lies in the “I will mow the grass” circle. But the converse, “If I mow the grass, then it is sunny,” is not necessarily true. There may be days when I cut the grass and it is not sunny, because not every point in the “I will mow the grass” circle is in the “it is sunny” circle.

Think about the contrapositive: “If I do not mow the grass, then it is not sunny.” This is true because if you are not in the outer circle, then there is no way you can be in the inner circle. Or, said another way, “If not q, then not p.”

**Logical Chains**

Euler diagrams are not limited to two circles. There can be several circles, forming a chain of events. The syllogism you saw earlier was a **logical chain**. It is an example of what mathematicians call the *If-Then Transitive Property*. Unlike syllogism, which is limited to two premises, the If-Then Transitive Property can have an unlimited number of premises.

**If-Then Transitive Property**  
If $p \rightarrow q$ and $q \rightarrow r$, then $p \rightarrow r$.

In Euler diagrams, the If-Then Transitive Property looks like this.

![Euler diagram](image)

If a point lies in the innermost circle, then it must lie in the middle and outer circles as well. Here are two examples of the If-Then Transitive Property. The first example has two premises; the second example has three premises.

If it is sunny, then I will mow the grass.  
If I mow the grass, then I will get paid $20. Therefore, if it is sunny, then I get paid $20.
If it is sunny, then I will mow the grass.
If I mow the grass, then I will get paid $20.
If I get paid $20, then I will buy a new hat.
Therefore, if it is sunny, then I will buy a new hat.

**Summary**

- A conditional statement is a statement in “if-then” form. What follows the word *if* is the hypothesis and what follows the word *then* is the conclusion.
- The converse of a conditional statement switches the hypothesis and the conclusion. The converse of a true conditional may or may not be true.
- The inverse of a conditional statement negates both the hypothesis and the conclusion. The inverse of a true conditional statement may or may not be true.
- The contrapositive switches and negates the hypothesis and the conclusion. The contrapositive of a true conditional statement is always true. Proving the contrapositive of a conditional statement also proves the conditional statement.
- Truth tables can be used to tell you whether a statement is true or false by way of filling in the truth value of each part of the statement.
- Euler diagrams show how the parts of conditional statements are related to each other.
- The If-Then Transitive Property allows you to create logical chains. It says if \( p \to q \) and \( q \to r \), then \( p \to r \).
Compound Statements and Indirect Proof

You are a lawyer representing Zenith Railways in a lawsuit filed by Donald Sleepwell, who claims he was injured in a train accident. Mr. Sleepwell says he received head injuries when the train stopped suddenly, causing his head to hit the wall while he was sleeping in his berth. Mr. Sleepwell has asked the court to order Zenith Railways to pay him $5 million for medical costs and loss of earning power at his job.

The porter and several passengers have testified that the sleeping berths had been set up so people slept with their heads facing the direction the train was going. But they also have testified that the train was traveling in reverse.

You argue that anyone lying in the berths would have been sent feet-first, not head-first, toward the wall when the brakes were slammed on. You win! Because you have shown that Mr. Sleepwell was not telling the truth, the court has ruled in favor of your client, Zenith Railways.

You have proven your case indirectly.

KEYWORDS
compound statement conjunction
contradiction disjunction
exclusive or inclusive or
indirect proof proof by contradiction

Compound Statements

Recall that a conditional statement contains a hypothesis and a conclusion. In all our examples so far, the hypothesis has been a single statement. A compound statement connects two statements with either the word and or the word or. If they are connected by and, then it is a conjunction. If they are connected by or, then it is a disjunction.
Conjunction: It is sunny and the temperature is greater than 90°F.
Disjunction: In baseball, a hitter is pitched four balls or he is hit by a pitch.

A compound conditional statement uses a compound statement as its hypothesis:

If it is sunny and the temperature is greater than 90°F, then I will go swimming.

In baseball, if a hitter is pitched four balls or he is hit by a pitch, then he can go to first base.

**Truth Tables for Compound Statements**

You can use truth tables to determine if a compound statement is true or false. The symbol $\wedge$ means *and*; the symbol $\lor$ means *or*.

**And**  A conjunction is true only when $p$ and $q$ are both true. Otherwise, the compound statement is false.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \wedge q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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</tbody>
</table>

**Or**   A disjunction is false only when $p$ and $q$ are both false. Otherwise, the compound statement is true.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \lor q$</th>
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</thead>
<tbody>
<tr>
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</tbody>
</table>

The disjunction just shown is the *inclusive or*. It includes the case where both $p$ and $q$ are true. The *exclusive or* excludes the case where both $p$ and $q$ are true. Mathematicians use the inclusive or, which is what we will use in this book.
**Indirect Proofs**

An **indirect proof** is the first type of formal proof we will study. In an indirect proof, we start by assuming the opposite of what we are trying to prove. Then we use deductive reasoning to reach a contradiction. A **contradiction** is a statement that disagrees with another statement. The contradiction will show that the assumption was false and therefore what we wanted to prove must be true. The contradiction can contradict either the given, a definition, a postulate, a theorem, or any known fact. Because an indirect proof uses a contradiction, we can also call this type of proof a **proof by contradiction**.

**Given**  
Triangle $\triangle ABC$

**Prove**  
A triangle cannot have two right angles.

**Indirect Proof**

Assume triangle $\triangle ABC$ has two right angles: $\angle B$ and $\angle C$. Because the measures of the angles of a triangle always sum to $180^\circ$, $m\angle A + m\angle B + m\angle C = 180^\circ$. Because a right angle measures $90^\circ$, substitute $90^\circ$ for $m\angle B$ and $m\angle C$. So, $m\angle A + 90^\circ + 90^\circ = 180^\circ$. Simplifying the left side results in $m\angle A + 180^\circ = 180^\circ$. Solving for $m\angle A$ gives $m\angle A = 0^\circ$. But if this is true, then $AB$ would lie on top of $AC$, and the figure would not be a triangle. This contradicts the given information. The assumption that a triangle has two right angles is false. Therefore, a triangle cannot have two right angles.

**Summary**

- A compound statement combines two statements with either the word *and* or the word *or*. A conjunction combines two statements with the word *and*. A disjunction combines two statements with the word *or*.
- The inclusive use of *or* allows both statements to be true at the same time. If both statements cannot be true at the same time, the exclusive use of *or* is being used.
- One way to prove a statement is with an indirect proof. An indirect proof assumes the opposite of what you are trying to prove and uses a contradiction to prove the original statement true.
Definitions and Biconditionals

Definitions must be clear and precise. Suppose you define the word *sunflower* as “a flower with yellow petals.” Does it then follow that every flower with yellow petals is a sunflower? Roses, marigolds, and tulips also can have yellow petals, so the answer is No. The same problem can happen in mathematics. You need good definitions that will correctly identify or classify objects.

**Definitions**

If a statement is a definition, then both the original statement and its converse must be true. A statement that is not written as a conditional can be rewritten in the conditional form.

You have learned that a right angle measures $90^\circ$. This can be written as: *If an angle is a right angle, then it measures $90^\circ$.* In this case the converse is also true: *If an angle measures $90^\circ$, then it is a right angle.*

**Adjacent angles** are two coplanar angles that share a common side and have no interior points in common. As this is a definition, both the statement and its converse must be true.

**Statement**  If two angles are adjacent, then the angles are coplanar, share a common side, and have no interior points in common.

**Converse**  If two angles are coplanar, share a common side, and have no interior points in common, then the angles are adjacent.
**Biconditional Statements**

If a statement and its converse are both true, then the statement is **biconditional**. A biconditional statement in mathematics may be written using the phrase *if and only if*, which is abbreviated like this: *iff*. The symbol $\leftrightarrow$ represents a biconditional statement, meaning $p$ implies $q$ and $q$ implies $p$.

The definition of adjacent angles can be rewritten as follows:

Two angles are adjacent if and only if they are coplanar, share a common side, and have no interior points in common.

**Euler Diagrams**

You can use an Euler diagram to determine if a statement is a definition. Suppose a car was defined as a vehicle with four wheels. Written as a conditional, the statement is: *If it is a car, then it is a vehicle with four wheels.* The Euler diagrams for the statement and its converse are shown.

![Euler Diagrams](image)

Look at the diagram for the converse. Sport utility vehicles, pickups, wheelchairs, and wagons would fall within the center circle, but not within the outer circle. The diagram does not make sense. The converse is not true; the statement is not a definition.

Look at a math definition: A quadrilateral is a polygon with four sides. The conditional and the converse are: *If a figure is a quadrilateral, then it has four sides* and *If a figure has four sides, then it is a quadrilateral*.

![Euler Diagrams](image)

The statement and its converse are both true, so you can rewrite the definition of a quadrilateral as: *A figure is a quadrilateral if and only if it is a polygon with four sides.*
Creating Definitions

Creating a definition is not always easy. If you define a square as a rectangle, then its converse is not true. The converse is: \textit{A rectangle is a square}. That statement isn’t accurate. Some rectangles aren’t squares. A square is a rectangle with consecutive sides congruent. Here are some tips for writing a good definition.

- Be precise. Avoid terms like \textit{sort of}.
- Use only previously defined terms.
- Do not use the word you are defining in the definition.
- Tell what the object is, not what the object is not.

Summary

- A statement is a definition if the statement and its converse are both true. We call them biconditional statements.
- A biconditional statement uses the phrase \textit{if and only if} to show that the statement and its converse are both true.
- A good definition has terms that are clear and that have been defined previously, makes it easy to determine the reasons as to why it is part of a certain classification, and uses as few words as possible.
Throughout this unit, you have explored the nature of, and some methods for, proof. The responsibility of supplying convincing evidence in support of your statement or issue rests with you. Just as a competent attorney studies his or her case carefully and thoroughly, you need to investigate your specific assignment completely from every viewpoint. Draw pictures and diagrams to visualize the conditions at hand; build a comprehensive algebraic and geometric “arsenal” of definitions and properties to help you list your options; and be willing to make intelligent, sensible guesses knowing that you may come to a dead end and may have to try another route.

**KEYWORDS**
- conjecture
- theorem
- direct proof
- two-column proof

**Properties of Equality**

**Algebraic Properties of Equality**
- **The Addition Property of Equality**
  \[ a + c = b + c. \]
- **The Subtraction Property of Equality**
  \[ a - c = b - c. \]
- **The Multiplication Property of Equality**
  \[ ac = bc. \]
- **The Division Property of Equality**
  \[ \frac{a}{c} = \frac{b}{c}. \]
- **The Substitution Property of Equality**
  If \( a = b \), then \( b \) can be substituted for \( a \) in any expression.

**Equivalence Properties of Equality and Congruence**
- **The Reflexive Property of Equality**
  \[ a = a \]
- **The Reflexive Property of Congruence**
  \[ a \cong a \]
- **The Symmetric Property of Equality**
  If \( a = b \), then \( b = a \).
- **The Symmetric Property of Congruence**
  If \( a \cong b \), then \( b \cong a \).
- **The Transitive Property of Equality**
  If \( a = b \) and \( b = c \), then \( a = c \).
- **The Transitive Property of Congruence**
  If \( a \cong b \) and \( b \cong c \), then \( a \cong c \).

**MEMORY TIP**

_Reflective_ sounds like _reflecting_. A reflection looks exactly the same as the original object.
**Theorems and Proofs**

A theorem is a mathematical statement that has been or is to be proven on the basis of established definitions and properties. A major way of proving theorems is the method of direct proof, in which the conclusion is drawn directly from previous conclusions, starting with the first statement. Also, geometric proofs can be written in two basic formats: two-column or paragraph. A paragraph proof is written in sentences and is more difficult than the two-column proof, which is easier to set up and understand.

The statement that is to be proven is called a conjecture. A conjecture is a statement you think is true, but still need to prove using deductive reasoning. Once a conjecture is proven, it is called a theorem. You can use inductive reasoning to come up with conjectures, then use deductive reasoning to prove or disprove your conjectures.

A two-column geometric proof uses deductive reasoning and consists of a list of statements, each with the formal reason (the given, definition, postulate, theorem already proven, or property) that justifies how you know that the statement is true. The statements are listed in a column on the left, and the reasons for which the statements can be made are listed in a column on the right. Every step of the proof (a subconclusion) is a row in the two-column proof.

Some guidelines for writing proofs are as follows:

- Draw the figure that illustrates what is to be proven. The figure may have already been drawn for you, or you may have to draw it yourself.
- List the given statements, and then list the conclusion to be proven. Now you have a beginning and an end to the proof.
- Mark the figure according to what you can conclude about it from the information that is given. This is the step of the proof in which you actually find out how the proof is to be made, and whether you are able to prove what is asked. Mark up all that you are able so that you can see for yourself what you must write in your proof to convince the reader that your conclusion is valid and accurate.
- Write your steps carefully—even the simplest ones. Some of the first steps are often the given statements (but not always), and the last step is the conclusion that is to be proven.
Theorems 2-1 and 2-2 are similar. Study their proofs carefully.

**THEOREM 2-1**  
**Theorem of Overlapping Segments**  
*If point \( B \) is between \( A \) and \( C \) and point \( C \) is between \( B \) and \( D \), then \( AB = CD \) iff \( AC = BD \).*

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( AB = CD )</td>
<td>Given</td>
</tr>
<tr>
<td>2. ( AB + BC = BC + CD )</td>
<td>Addition Property of Equality</td>
</tr>
<tr>
<td>3. ( AB + BC = AC )</td>
<td>Segment Addition Postulate</td>
</tr>
<tr>
<td>4. ( BC + CD = BD )</td>
<td>Segment Addition Postulate</td>
</tr>
</tbody>
</table>
| 5. \( AC = BD \) | Substitution Property of Equality  
(Steps 2 and 3, then 2 and 4) |

**THEOREM 2-2**  
**Theorem of Overlapping Angles**  
*If point \( B \) lies in the interior of \( \angle AWC \) and point \( C \) lies in the interior of \( \angle BWD \), then \( m\angle AWB = m\angle CWD \) iff \( m\angle AWC = m\angle BWD \).*

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( m\angle AWB = m\angle CWD )</td>
<td>Given</td>
</tr>
<tr>
<td>2. ( m\angle AWB + m\angle BWC = m\angle BWC + m\angle CWD )</td>
<td>Addition Property of Equality</td>
</tr>
<tr>
<td>3. ( m\angle AWB + m\angle BWC = m\angle AWC )</td>
<td>Angle Addition Postulate</td>
</tr>
<tr>
<td>4. ( m\angle BWC + m\angle CWD = m\angle BWD )</td>
<td>Angle Addition Postulate</td>
</tr>
</tbody>
</table>
| 5. \( m\angle AWC = m\angle BWD \) | Substitution Property of Equality  
(Steps 2 and 3, then 2 and 4) |

Sometimes this book will show a third column with a sketch to illustrate each reason. This proof will still be considered a two-column proof.
Summary

- One way to display a proof of a theorem is to use a two-column format.
- In a two-column proof, statements are presented in a list in a column on the left, and reasons are presented in a column on the right. Reasons can include the given, definitions, postulates, theorems already proven, and properties.
Your five-year-old sister notices that every time she throws a ball up into the air, it comes down and hits her on the head. You tell her to do it again, but she says: “No way. I will get hit on the head again!” She is thinking inductively. Your 15-year-old brother says: “That is Newton’s law of gravity—what goes up must come down.” He is reasoning deductively. Briefly, induction moves from the specific to the general, while deduction progresses from the general to the specific. Assertions that are built on observation and experience are inductive, and reasoning that is based on established principles is deduction. The distinction between the two is the point of view. Remember that deductive reasoning is proof and that observation, measurement, and experimentation are not proof.

**KEYWORDS**
- inductive reasoning
- paragraph proof
- vertical angles

**Using Inductive Reasoning**

You may recall from Unit 1 that a conjecture is an educated guess. Making a conjecture based on past events or on a pattern is called **inductive reasoning**. Although curiosity, observation, and conjecture have an important role in the proof process, inductive reasoning sometimes fails. Inductive reasoning provides us with a starting point, but it is not proof.

You can use geometry software to emphasize the inductive process, as shown by the following two examples. They demonstrate the process of inductive reasoning, combined with the use of algebra to express geometric ideas.

*A reflection across a line and then across a line parallel to the first line is equivalent to a translation of twice the distance between the lines and in a direction perpendicular to the lines.*

In this sketch, the triangle was first reflected across the red dashed line. Then it was reflected across the solid blue line. The distance between the red dashed and solid blue lines is two units. The final image is four units away from the original pre-image.
**Vertical Angles**

*Vertical angles* are the two nonadjacent angles formed by intersecting lines. The sides of one angle are the opposite rays to the sides of the other angle. In the diagram, \( \angle 1 \) and \( \angle 3 \) form a vertical pair, and \( \angle 2 \) and \( \angle 4 \) form another vertical pair.

We can prove that vertical angles are congruent by means of a two-column proof.

**THEOREM 2-3  Vertical Angles Theorem**
If two angles form a pair of vertical angles, then they are congruent.

<table>
<thead>
<tr>
<th>Given</th>
<th>( \angle 1 ) and ( \angle 3 ) are vertical angles.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prove</td>
<td>( \angle 1 \cong \angle 3 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \angle 1 ) and ( \angle 3 ) are vertical angles.</td>
<td>Given</td>
</tr>
<tr>
<td>2. ( \angle 1 ) and ( \angle 2 ) form a linear pair. ( \angle 2 ) and ( \angle 3 ) form a linear pair.</td>
<td>Definition of linear pair</td>
</tr>
<tr>
<td>3. ( m\angle 1 + m\angle 2 = 180^\circ ) ( m\angle 2 + m\angle 3 = 180^\circ )</td>
<td>Linear Pair Postulate</td>
</tr>
<tr>
<td>4. ( m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3 )</td>
<td>Substitution Property of Equality</td>
</tr>
<tr>
<td>5. ( m\angle 1 = m\angle 3 )</td>
<td>Subtraction Property of Equality</td>
</tr>
<tr>
<td>6. ( \angle 1 \cong \angle 3 )</td>
<td>Angle Congruence Postulate</td>
</tr>
</tbody>
</table>

You can also write a proof in the form of a paragraph. This is called a **paragraph proof**. A paragraph proof for the Vertical Angles Theorem is shown below. Notice that, similar to the two-column proof, each statement is supported by a reason.

**Given** \( \angle 1 \) and \( \angle 3 \) are vertical angles.

**Prove** \( \angle 1 \cong \angle 3 \)

You are given that \( \angle 1 \) and \( \angle 3 \) are vertical angles. Because of the definition of *linear pair*, \( \angle 1 \) and \( \angle 2 \) form a linear pair and \( \angle 2 \) and \( \angle 3 \) form a linear pair. By the Linear Pair Postulate, \( m\angle 1 + m\angle 2 = 180^\circ \) and \( m\angle 2 + m\angle 3 = 180^\circ \). By using the Substitution Property of Equality...
and substituting $m\angle 2 + m\angle 3$ for 180° into the first equation, we get $m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$. By using the Subtraction Property of Equality, we can subtract $m\angle 2$ from both sides to get $m\angle 1 = m\angle 3$. Because of the definition of congruent, $\angle 1 \cong \angle 3$.

Most of the time, we will use the two-column format because it is easier to set up and to understand.

<table>
<thead>
<tr>
<th>Proof of $\angle 1 \cong \angle 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Given</strong> $\angle 2 \cong \angle 3$</td>
</tr>
<tr>
<td>$\angle 1 \cong \angle 2$</td>
</tr>
<tr>
<td>$\angle 1 \cong \angle 3$</td>
</tr>
<tr>
<td>$\angle 3 \cong \angle 4$</td>
</tr>
</tbody>
</table>

You can find the measures of all four angles formed by a pair of intersecting lines when you are given just one of the measures.

In the diagram, $m\angle 3 = 20^\circ$.

Because $\angle 2$ and $\angle 3$ form a linear pair, then $m\angle 2 = 160^\circ$.

Because $\angle 2$ and $\angle 4$ are vertical angles, $m\angle 4 = 160^\circ$.

Because $\angle 1$ and $\angle 3$ are vertical angles, $m\angle 1 = 20^\circ$.

There are several ways to find the measures of the three angles that are not given. For example, you could have also used the Linear Pair Postulate all around the figure. Similarly, you will find there are sometimes multiple ways to prove a statement. As long as each statement follows logically from previous statements, the proof is valid.
Summary

• Inductive reasoning is based on observations of patterns and past events. Inductive reasoning can be used either to make or to support conjectures. It cannot be used to formally prove conjectures.

• A reflection across a line and then across a line parallel to the first line is equivalent to a translation of twice the distance between the lines and in a direction perpendicular to the lines.

• A reflection across two intersecting lines is equivalent to a rotation about the point of intersection through twice the measure of the angle between the lines.

• When two lines intersect, vertical angles are formed. Vertical angles are the nonadjacent angles whose sides are opposite rays.

• The Vertical Angles Theorem states that vertical angles are congruent.

• A paragraph proof is another type of formal proof. Each statement is supported by a reason as in the two-column proof.