Antoni Gaudi used parabolic arches in his design of Park Guell.
Antoni Gaudi (1852–1926) used parabolic arches in his architectural designs. Gaudi’s parabolic arches are both beautiful and strong.

Big Ideas

► Expressions, equations, and inequalities express relationships between different entities.

► If you use a mathematical model to represent a certain situation, you can use the model to solve other problems that you might not be able to solve otherwise. Algebraic equations can capture key relationships among quantities in the world.

► A function is a correspondence between two sets, the domain and the range, that assigns to each member of the domain exactly one member of the range. Many functions can be described by algebraic expressions.

Unit Topics

► Foundations for Unit 8
► Graphing Quadratic Functions
► Properties of Quadratic Graphs
► Solving Quadratic Equations
► Quadratic Inequalities
► Finding a Quadratic Function from Points
► Applications: Quadratic Functions
Foundations for Unit 8

Before you study quadratic functions, you should know how to do the following:

- Determine lines of symmetry from a figure.
- Understand function notation and evaluate a function.
- Find the equation of an absolute value function from a graph.

Identifying Lines of Symmetry

Definitions

line symmetry  a figure is said to have line symmetry (or reflection symmetry) if there is at least one line such that when the figure is folded over the line, the two halves are mirror images that match up perfectly.

line of symmetry  the line over which you can flip a given figure, leaving the figure unchanged; the line that divides a given figure into two congruent (mirror-image) halves.

Example 1  Determine the equations of the lines of symmetry (if any) for each figure.

A

The four lines of symmetry are \( y = 0, y = -x, x = 0, \) and \( y = x. \)

B

The two lines of symmetry are \( y = 1 \) and \( x = 2. \)

C

The figure has no lines of symmetry.

TIP

If you fold the graph over a line of symmetry, the two halves will match.
Problem Set A

Determine the equations of the lines of symmetry (if any) for each figure.

1. 

2. 

3. 

4. 

Using Function Notation and Evaluating a Function

FUNCTION NOTATIONS FOR $y = x^2$

$f: \text{input} \rightarrow \text{output}$

$f(x) \rightarrow x^2$

REMEMBER

A function is a relation in which every element of the domain is assigned to exactly one element of the range.

Example 2  Evaluate each function for the given values.

A  For $f(x) = 3x - 7$, find $f(2)$ and $f(5)$.

Substitute 2 and 5 for $x$.

$f(2) = 3 \cdot 2 - 7 = 6 - 7 = -1$

$f(5) = 3 \cdot 5 - 7 = 15 - 7 = 8$

B  For $g: a \rightarrow -a^2$, find $g(3)$ and $g(-3)$.

Substitute 3 and $-3$ for $a$.

$g(3) = -3^2 = 9$

$g(-3) = -(-3)^2 = -9$
Problem Set B

Evaluate each function for the given value.

5. For \( f(x) = 2x + 5 \), find \( f(2) \).
6. For \( h: t \to 2t^2 + 1 \), find \( h(7) \) and \( h(-7) \).
7. For \( f(x) = (x - 3)^2 \), find \( f(4) \) and \( f(2) \).
8. For \( g: b \to b^3 \), find \( g(2) \) and \( g(-2) \).
9. For \( h: x \to 2x^3 + 3 \), find \( h(1) \).
10. For \( f(z) = 1 - z^3 \), find \( f(1) \) and \( f(4) \).

Finding an Absolute Value Equation when Given a Graph

For an absolute value function, the parent function is \( f(x) = |x| \) and the general form for functions in this family is \( f(x) = a|x - h| + k \), where the point \((h, k)\) is the coordinate of the highest or lowest point on the graph, and where \(a\) determines how wide or narrow the graph is and whether it opens up or down.

Example 3  Find an absolute value equation for each graph.

The highest point is \((-2, 3)\). The horizontal and vertical translations from the graph of the parent function, \( f(x) = |x| \), reveal that the equation of the graph is \( y = -|x + 2| + 3 \).

The lowest point is \((1, 4)\). The horizontal and vertical translations from the graph of the parent function, \( f(x) = |x| \), reveal that the equation of the graph is \( y = |x - 1| + 4 \).

Problem Set C

Find an absolute value equation for each graph.

11.
12.
13.
Graphing Quadratic Functions

A quadratic function is a second-degree polynomial function.

**STANDARD FORM OF A QUADRATIC FUNCTION**

The standard form of a quadratic function is \( f(x) = ax^2 + bx + c \), where \( a \neq 0 \). The graph of a quadratic function is called a parabola.

One way to graph a quadratic function is to create a table of ordered pairs, plot the points, and then draw a smooth curve through those points. As you study the following examples, recall that \( y \) and \( f(x) \) are used interchangeably.

**Using a Table to Graph a Quadratic Function**

**Example 1** Graph each function.

A. \( y = x^2 \)

**Solution**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( -2 )</th>
<th>( -1 )</th>
<th>( 0 )</th>
<th>( 1 )</th>
<th>( 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

The point \((-2, -8)\) is not shown.

B. \( y = -2x^2 + 2x + 4 \)

**Solution**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( -2 )</th>
<th>( -1 )</th>
<th>( 0 )</th>
<th>( 1 )</th>
<th>( 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>-8</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>

The point \((-2, -8)\) is not shown.
Graphing Quadratic Functions in Standard Form

All quadratic function graphs have in common the following characteristics, which can be used to create accurate graphs.

**PROPERTIES OF QUADRATIC FUNCTION GRAPHS**

The graph of \( f(x) = ax^2 + bx + c \) is a parabola with these characteristics:

- It opens up when \( a > 0 \); it opens down when \( a < 0 \).
- It has \( y \)-intercept \( c \), so \((0, c)\) is a point on the graph.
- It has a vertex with \( x \)-coordinate \(- \frac{b}{2a}\).
- It has an axis of symmetry with equation \( x = - \frac{b}{2a} \). The axis of symmetry is the vertical line through the vertex. It separates the graph into two halves that are reflections (mirror images) of each other.

**Example 2**  Graph \( y = 3x^2 - 12x + 6 \).

**Solution**  First identify the coefficients \( a, b, \) and \( c \):

\[
\begin{align*}
  y &= 3x^2 - 12x + 6, \\
  a &= 3, \\
  b &= -12, \\
  c &= 6
\end{align*}
\]

The leading coefficient \( a \) is positive, so the parabola opens up, not down.

The \( x \)-coordinate of the vertex is \(- \frac{b}{2a} = - \frac{(-12)}{2 \cdot 3} = 2\).

To find the \( y \)-coordinate of the vertex, substitute 2 for \( x \):

\[
\begin{align*}
  y &= 3x^2 - 12x + 6 \\
  &= 3(2)^2 - 12 \cdot 2 + 6 \\
  &= -6.
\end{align*}
\]

So the vertex \( V \) is \((2, -6)\), and the equation of the axis of symmetry is \( x = 2 \).

The \( y \)-intercept is \( c = 6 \), so \((0, 6)\) is a point on the graph. The reflection image of \((0, 6)\) over the axis of symmetry is \((4, 6)\), which is also on the graph.

Choose any value for \( x \) and substitute it to find one more point on the parabola. If \( x = 1 \), then \( y = 3 \cdot 1^2 - 12 \cdot 1 + 6 = -3 \). So \((1, -3)\) is on the graph, and its reflection image \((3, -3)\) is also on the graph.

Draw a smooth curve through all five points, including the vertex.

**Determining the Number of Zeros of a Quadratic Function**

Using properties of quadratic functions can help you determine where the graph of a given function crosses the \( x \)-axis.

**ZEROS OF A POLYNOMIAL FUNCTION**

The zeros of a polynomial function \( f(x) \) are the roots (solutions) of the equation \( f(x) = 0 \). The real zeros are the \( x \)-intercepts of the graph of \( f(x) \).
Number of Real Zeros of a Quadratic Function

A quadratic function can have two, one, or no real zeros.

Example 3  Determine the number of real zeros of each quadratic function.

A  \( y = x^2 + 6x + 9 \)

**Solution**  \( a = 1, b = 6, \) and \( c = 9. \) Since \( a > 0, \) the parabola opens up.

Use \( a \) and \( b \) to find the \( x \)-coordinate of the vertex, and then substitute to find the \( y \)-coordinate:

\[
x = -\frac{b}{2a} = -\frac{6}{2 \cdot 1} = -3
\]

\[
y = x^2 + 6x + 9 = (-3)^2 + 6 \cdot (-3) + 9 = 9 - 18 + 9 = 0
\]

The vertex is \((-3, 0)\). The vertex is a zero of the function and because the parabola opens up, the vertex is also the lowest point on the graph. Therefore, it is the only real zero. Sketch the graph to verify this.

B  \( y = x^2 + 2 \)

**Solution**  \( a = 1, b = 0, \) and \( c = 2. \) Since \( a > 0, \) the parabola opens up.

Use \( a \) and \( b \) to find the \( x \)-coordinate of the vertex, and then substitute to find the \( y \)-coordinate:

\[
x = -\frac{b}{2a} = -\frac{0}{2 \cdot 1} = 0
\]

\[
y = x^2 + 2 = 0^2 + 2 = 2
\]

The vertex is \((0, 2)\). Since the parabola opens up, the vertex is the lowest point on the graph. Therefore, the function has no real zeros. Sketch the graph to verify this.

C  \( y = -x^2 + 4x - 1 \)

**Solution**  \( a = -1, b = 4, \) and \( c = -1. \) Since \( a < 0, \) the parabola opens down.

Use \( a \) and \( b \) to find the \( x \)-coordinate of the vertex, and then substitute to find the \( y \)-coordinate:

\[
x = -\frac{b}{2a} = -\frac{4}{2 \cdot (-1)} = 2
\]

\[
y = -x^2 + 4x - 1 = -2^2 + 4 \cdot 2 - 1 = -4 + 8 - 1 = 3
\]

The vertex is \((2, 3)\). Since the parabola opens down, the vertex is the highest point. The graph must cross the \( x \)-axis at two points; therefore, the function has two real zeros. Sketch the graph to verify this.
Graphing Quadratic Functions in Factored Form

Using the factored form of a quadratic function can help you graph the function.

**FACTORED FORM OF A QUADRATIC FUNCTION**

The factored form of a quadratic function is \( f(x) = a(x - r_1)(x - r_2) \).

The graph of \( f(x) = a(x - r_1)(x - r_2) \) is a parabola with these characteristics:

- It opens up if \( a > 0 \); it opens down if \( a < 0 \).
- It has **x-intercepts** \( r_1 \) and \( r_2 \).
- It has an **axis of symmetry** with equation \( x = \frac{r_1 + r_2}{2} \) (halfway between the x-intercepts).

**Example 4** Use the factored form to graph each function.

**A** \( y = x^2 - 2x - 3 \)

**Solution** Factor the trinomial: \( y = x^2 - 2x - 3 = (x + 1)(x - 3) \).

\( a = 1 \), so the parabola opens up.

\( r_1 = -1 \) and \( r_2 = 3 \), so the x-intercepts are \(-1\) and \(3\).

\[ \frac{r_1 + r_2}{2} = \frac{-1 + 3}{2} = \frac{2}{2} = 1 \]

So the equation of the axis of symmetry is \( x = 1 \). Since the axis of symmetry passes through the vertex, the x-coordinate of the vertex must be 1. Substitute 1 for \( x \) in the function to find the y-coordinate of the vertex:

\[ y = x^2 - 2x - 3 = 1^2 - 2 \cdot 1 - 3 = 1 - 2 - 3 = -4 \]

So the vertex \( V \) is \((1, -4)\).

Find two more points, such as \((0, -3)\) and its reflection, \((2, -3)\). Then sketch the curve through all the points.

**B** \( y = -0.4x^2 + 0.4x + 4.8 \)

**Solution** Factor out \(-0.4\) and then factor the trinomial:

\[ y = -0.4x^2 + 0.4x + 4.8 = -0.4(x^2 - x - 12) = -0.4(x - 4)(x + 3) \]

\( a = -0.4 \), so the parabola opens down.

\( r_1 = 4 \) and \( r_2 = -3 \), so the x-intercepts are 4 and \(-3\).

\[ \frac{r_1 + r_2}{2} = \frac{4 + (-3)}{2} = \frac{1}{2} = 0.5 \]

So the equation of the axis of symmetry is \( x = 0.5 \). Since the axis of symmetry passes through the vertex, the x-coordinate of the vertex must be 0.5. Substitute 0.5 for \( x \) in the function to find the y-coordinate of the vertex:

\[ y = -0.4(x - 4)(x + 3) = -0.4(0.5 - 4)(0.5 + 3) = 4.9 \]

So the vertex \( V \) is \((0.5, 4.9)\).

Find two more points, such as \((2, 4)\) and its reflection, \((-1, 4)\). Then sketch the curve through all the points.
**Problem Set**

For each function, do the following:

A. Complete the table of values.
B. Use the table of values to graph the quadratic equation.

1. \( y = x^2 - 2x \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. \( y = x^2 + 4x \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. \( y = x^2 + 2 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. \( y = -x^2 + 3 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. \( y = x^2 + 3x - 4 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6. \( y = x^2 - 5x + 7 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Find the y-intercept of each quadratic function.

7. \( y = -x^2 - 3 \)
8. \( y = 3x^2 - 4 \)
9. \( y = x^2 + 5x - 8 \)
10. \( y = -2x^2 + 5x \)

Graph each quadratic function. For each graph, label the vertex, x-intercepts, and the equation of the axis of symmetry.

11. \( y = -x^2 + 1 \)
12. \( y = 2x^2 \)
13. \( y = 7x^2 - 7 \)
14. \( y = -3x^2 + 4x - 2 \)

Find the number of zeros for each quadratic function. Use a graph to check your answer.

15. \( y = x^2 - 8x \)
16. \( y = x^2 + 2x + 1 \)
17. \( y = 6x^2 + 8 \)
18. \( y = x^2 - 8x + 16 \)
19. \( y = x^2 + 6x + 8 \)
20. \( y = -4x^2 + 5x \)
Graph each function

21. \( y = (x + 1)(x - 3) \)
22. \( y = (x + 6)(x - 1) \)
23. \( y = -x^2 - x + 20 \)
24. \( y = 3x^2 + 6x - 72 \)
25. \( y = -2x^2 + 6x + 56 \)
26. \( y = -5x^2 - 25x - 30 \)
27. \( y = \frac{1}{8}x^2 + \frac{1}{24}x - \frac{5}{96} \)
*28. **Challenge** \( y = -1.3x^2 + 3.9x - 5.2 \)
*29. **Challenge** \( y = \frac{1}{2}x^2 - 18.4x + 4 \)

Solve.

30. The cost \( C \), in dollars, of building \( m \) sewing machines at Sienna’s Sewing Machines is given by the equation \( C(m) = 20m^2 - 830m + 15,000 \).
   A. Find the cost of building 75 sewing machines.
   B. How many sewing machines should the company manufacture to minimize the cost \( C \)?
Properties of Quadratic Graphs

Just as there are different forms for the equation of a linear function, there are different forms for the equation of a quadratic function.

Quadratic Functions in Vertex Form

The vertex form of a quadratic function is
\[ f(x) = a(x - h)^2 + k, \]
where \( a \neq 0 \).

The vertex is \((h, k)\). The equation of the axis of symmetry is \( x = h \).

If \( a > 0 \), the graph opens up, extending infinitely up, left, and right. The minimum function value is \( k \). There is no maximum function value.

If \( a < 0 \), the graph opens down, extending infinitely down, left, and right. The maximum function value is \( k \). There is no minimum function value.

Example 1

Graph \( f(x) = \frac{1}{2}(x - 4)^2 - 2 \).

Solution

\( h = 4 \) and \( k = -2 \), so the vertex is \((4, -2)\) and the line of symmetry is \( x = 4 \). Find two other points on the graph.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \frac{1}{2}(x - 4)^2 - 2 )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>( \frac{1}{2}(5 - 4)^2 - 2 = \frac{1}{2} \cdot 1 - 2 = -1 \frac{1}{2} )</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>( \frac{1}{2}(6 - 4)^2 - 2 = \frac{1}{2} \cdot 4 - 2 = 0 )</td>
<td></td>
</tr>
</tbody>
</table>

Reflect the points \((5, -1.5)\) and \((6, 0)\) across the axis of symmetry to get two points on the other side: \((3, -1.5)\) and \((2, 0)\). Draw a smooth curve through all the points.
Writing an Equation of a Quadratic Function, Given a Graph Showing the Vertex and Another Point

Example 2  Write an equation for each quadratic function graphed.

**Solution**  The vertex \((h, k) = (0, 3)\), so \(h = 0\) and \(k = 3\).

\[ f(x) = a(x - h)^2 + k \quad \text{Vertex form} \]
\[ f(x) = a(x - 0)^2 + 3 \quad \text{Substitute 0 for } h \text{ and 3 for } k. \]
\[ f(x) = ax^2 + 3 \quad \text{Simplify.} \]
\[ -1 = a \cdot 2^2 + 3 \quad \text{Substitute 2 for } x \text{ and } -1 \text{ for } y. \]
\[ -1 = 4a + 3 \quad \text{Simplify.} \]
\[ -4 = 4a \quad \text{Solve for } a. \]
\[ -1 = a \]

So the equation is \(f(x) = -x^2 + 3\). ■

**Solution**  The vertex \((h, k) = (-1, -3)\), so \(h = -1\) and \(k = -3\).

\[ f(x) = a(x - h)^2 + k \quad \text{Vertex form} \]
\[ f(x) = a(x - (-1))^2 - 3 \quad \text{Substitute } -1 \text{ for } h \text{ and } -3 \text{ for } k. \]
\[ f(x) = a(x + 1)^2 - 3 \quad \text{Simplify.} \]
\[ 5 = a(1 + 1)^2 - 3 \quad \text{Substitute 1 for } x \text{ and 5 for } y. \]
\[ 5 = a \cdot 2^2 - 3 \quad \text{Simplify.} \]
\[ 5 = 4a - 3 \quad \text{Solve for } a. \]
\[ 8 = 4a \]
\[ 2 = a \]

So the equation is \(f(x) = 2(x + 1)^2 - 3\). ■

Converting from Vertex to Standard Form

By using the FOIL method and collecting like terms, you can convert an equation given in vertex form into standard form.

Example 3  Write \(y = 3(x + 4)^2 - 8\) in standard form.

**Solution**

\[ y = 3(x + 4)^2 - 8 \quad \text{Original equation} \]
\[ y = 3(x + 4)(x + 4) - 8 \quad \text{Rewrite the exponent as multiplication.} \]
\[ y = 3(x^2 + 8x + 16) - 8 \quad \text{Use the FOIL method to multiply the binomials.} \]
\[ y = 3x^2 + 24x + 48 - 8 \quad \text{Distribute the 3.} \]
\[ y = 3x^2 + 24x + 40 \quad \text{Simplify.} \]

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Converting from Standard to Vertex Form

Converting an equation from standard to vertex form is a bit tougher than converting the other way, but if you remember how to complete the square, you can get it done.

**Definition**

**Completing the square** is the process of transforming an expression of the form \(x^2 + bx\) into a perfect square trinomial by adding the term \(\left(\frac{b}{2}\right)^2\) to it.

**Example 4** Write each function in vertex form by completing the square.

**A** \(f(x) = x^2 + 8x + 9\)

**Solution**

\[
\begin{align*}
  f(x) &= x^2 + 8x + 9 \\
  f(x) &= (x^2 + 8x + 16 - 16) + 9 \\
  f(x) &= (x + 4)^2 - 7
\end{align*}
\]

The function in vertex form is \(f(x) = (x + 4)^2 - 7\).

**B** \(y = -3x^2 + 12x + 5\)

**Solution**

\[
\begin{align*}
  y &= -3x^2 + 12x + 5 \\
  y &= -3(x^2 - 4x) + 5 \\
  y &= -3(x^2 - 4x + 4 - 4) + 5 \\
  y &= -3(x - 2)^2 + 12 + 5 \\
  y &= -3(x - 2)^2 + 17
\end{align*}
\]

The function in vertex form is \(y = -3(x - 2)^2 + 17\).

**Quadratic Function Graph Family:** \(f(x) = a(x - h)^2 + k\), where \(a \neq 0\)

The graph of \(f(x) = a(x - h)^2 + k\), \(a \neq 0\), is a parabola with vertex \((h, k)\) and axis of symmetry \(x = h\). The parent graph is \(f(x) = x^2\).

Changing \(a\) determines how wide or narrow the parabola is and whether it opens up or down.

Changing \(h\) moves the vertex left or right.

Changing \(k\) moves the vertex up or down.
Problem Set

Graph each function. Label at least three points on each graph.

1. \( f(x) = (x - 4)^2 \)
2. \( g(x) = (x - 5)^2 + 4 \)
3. \( f(x) = 3(x - 7)^2 + 6 \)
4. \( h(x) = 2(x - 3)^2 + 12 \)
5. \( g(x) = -6(x - 2)^2 + 9 \)
6. \( h(x) = \frac{1}{2}(x - 4)^2 + 1 \)

Write an equation for each quadratic function graphed.

7. \( \quad \)
8. \( \quad \)
9. \( \quad \)
10. \( \quad \)
11. \( \quad \)
12. \( \quad \)
13. \( \quad \)
14. \( \quad \)
15. \( \quad \)
16. \( \quad \)

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Convert each equation to vertex form.

17. \( y = -(x - 4.5)^2 + 6.25 \)  
18. \( y = 4(x - 2)^2 + 11 \)

Convert each equation to vertex form.

23. \( y = x^2 - 4x + 4 \)  
24. \( y = 9x^2 + 15x + \frac{25}{4} \)  
25. \( y = x^2 - 8x + 10 \)

Solve.

29. **Challenge** A baseball player hits a baseball in the air. The height \( y \) in feet of the baseball \( t \) seconds after it is hit is given by this function:

\[ y = -16t^2 + 96t + 14 \]

Find the maximum height of the baseball.
A **quadratic equation** is a second-degree polynomial equation that can be written in the standard form $0 = ax^2 + bx + c$, where $a \neq 0$.

### Using a Graph to Describe Solutions of a Quadratic Equation

Recall that the zeros of a polynomial function $f(x)$ are the roots (solutions) of the equation $f(x) = 0$, and the real roots of $f(x) = 0$ are the $x$-intercepts of $f(x)$.

#### Example 1

A Use the graph of $f(x) = 3x^2 - 8x + 2$ to estimate the solutions of $f(x) = 0$.

**Solution** The $x$-intercepts are approximately 0.3 and 2.4. So the solutions of $0 = 3x^2 - 8x + 2$ are approximately 0.3 and 2.4.

**Check**

$f(x) = 3x^2 - 8x + 2$

- $f(0.3) = 3 \cdot 0.3^2 - 8 \cdot 0.3 + 2$
  - $= 0.27 - 2.4 + 2$
  - $= -0.13$
  - $\approx 0 ✓$

- $f(2.4) = 3 \cdot 2.4^2 - 8 \cdot 2.4 + 2$
  - $= 17.28 - 19.2 + 2$
  - $= 0.08$
  - $\approx 0 ✓$

B Use the graph of $y = -x^2 - 1$ to describe the solutions of $0 = -x^2 - 1$.

**Solution** There are no $x$-intercepts. So the equation $0 = -x^2 - 1$ has no real solutions.

---

**REMEMBER**

You can use a graphing calculator or software to graph a quadratic function and to estimate the $x$-intercepts.

**THINK ABOUT IT**

The equation $0 = -x^2 - 1$ in Example 1B has two *imaginary* solutions: $i$ and $-i$ (complex numbers $0 + i$ and $0 - i$). However, complex numbers have no representation in the real coordinate plane.
Solving a Quadratic Equation

Example 2  Solve and check each equation.

A \[ 2x^2 + 50 = -20x \]

B \[ 0 = 6x^2 - 13x - 5 \]

Solutions  Write each equation in standard form. Then factor and use the zero product property.

A \[ 2x^2 + 50 = -20x \]
\[ 2x^2 + 20x + 50 = 0 \]
\[ 2(x^2 + 10x + 25) = 0 \]
\[ 2(x + 5)(x + 5) = 0 \]
\[ x + 5 = 0 \] or \[ x + 5 = 0 \]
\[ x = -5 \] or \[ x = -5 \]

The solution set is \( \{-5\} \).

Check
\[ 2x^2 + 50 = -20x \]
\[ 0 = 6x^2 - 13x - 5 \]
\[ 0 = (2x - 5)(3x + 1) \]
\[ 2x - 5 = 0 \] or \[ 3x + 1 = 0 \]
\[ 2x = 5 \] \[ 3x = -1 \]
\[ x = \frac{5}{2} \] \[ x = -\frac{1}{3} \]

The check for \( -\frac{1}{3} \) is similar.

You can solve some quadratic equations by using the square root property.

Example 3  Solve and check: \((x + 5)^2 = 8\).

Solution  The equation \((x + 5)^2 = 8\) is already in the form \(x^2 = a\), with the variable expression \(x + 5\) instead of the variable \(x\).

\((x + 5)^2 = 8\)
\[ x + 5 = \pm\sqrt{8} \] Square Root Property
\[ x = -5 \pm \sqrt{8} \] Isolate \(x\).
\[ x = -5 \pm 2\sqrt{2} \] Simplify the radical.

The solution set is \(\{-5 + 2\sqrt{2}, -5 - 2\sqrt{2}\}\). The approximate solutions are \(-5 + 2\sqrt{2} \approx -2.17\) and \(-5 - 2\sqrt{2} \approx -7.83\).

Check  You can check the approximate solutions as follows:
\((-2.17 + 5)^2 \approx 8.0089 \approx 8,\) and \((-7.83 + 5)^2 \approx 8.0089 \approx 8.\)  ■

REMEMBER

The square root property states that for any real number \(a\), if \(x^2 = a\), then \(x = \pm\sqrt{a}\).
Solving a Quadratic Equation by Completing the Square

You can solve any quadratic equation by completing the square. To use this process, you need an expression of the form $x^2 + bx$.

**HOW TO COMPLETE THE SQUARE FOR A QUADRATIC EQUATION**

Starting with $x^2 + bx = c$, add $\left(\frac{b}{2}\right)^2$ to both sides to complete the square. $x^2 + bx + \left(\frac{b}{2}\right)^2$ is a perfect square because $x^2 + bx + \left(\frac{b}{2}\right)^2 = (x + \frac{b}{2})^2$.

**Example 4**  Complete the square to solve.

A  $x^2 - 2x + 5 = 0$

**Solution**

\[
x^2 - 2x + 5 = 0
\]

Subtract 5 to get the form $x^2 + bx$.

\[
x^2 - 2x = -5
\]

Add $\left(\frac{2}{2}\right)^2 = (1)^2 = 1$ to both sides to complete the square.

\[
(x - 1)^2 = -4
\]

Rewrite $x^2 - 2x + 1$ as $(x - 1)^2$.

\[
x - 1 = \pm\sqrt{-4}
\]

Square Root Property

\[
x = 1 \pm \sqrt{-4}
\]

Solve.

\[
x = 1 \pm 2i
\]

Simplify.

The solution set is $\{1 + 2i, 1 - 2i\}$. ■

B $9x = 56 - 2x^2$

**Solution**

\[
9x = 56 - 2x^2
\]

Add $2x^2$ to isolate the constant 56.

\[
2x^2 + 9x = 56
\]

Divide every term by 2 to get the form $x^2 + bx$.

\[
x^2 + \frac{9}{2}x = 28
\]

Add $\left(\frac{9}{4}\right)^2 = \left(\frac{1}{2}\cdot\frac{9}{2}\right)^2 = \left(\frac{9}{4}\right)^2$ to both sides to complete the square.

\[
\left(x + \frac{9}{4}\right)^2 = \frac{529}{16}
\]

Rewrite the trinomial $x^2 + \frac{9}{2}x + \left(\frac{9}{4}\right)^2$ as $\left(x + \frac{9}{4}\right)^2$.

\[
x + \frac{9}{4} = \pm\sqrt{\frac{529}{16}}
\]

Square Root Property

\[
x = -\frac{9}{4} \pm \frac{23}{4}
\]

Subtract and simplify.

The solution set is $\left\{\frac{7}{2}, -8\right\}$. ■

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Solving a Quadratic Equation by Using the Quadratic Formula

If you use the process of completing the square to solve the general quadratic equation \( ax^2 + bx + c = 0 \), you get a general formula for solving any quadratic equation.

\[
ax^2 + bx + c = 0 \\
ax^2 + bx = -c \\
x^2 + \frac{b}{a}x = -\frac{c}{a} \\
x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} \\
\left(x + \frac{b}{2a}\right)^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} \\
x + \frac{b}{2a} = \pm \sqrt{-\frac{b^2 - 4ac}{4a^2}} \\
x = -\frac{b}{2a} \pm \sqrt{-\frac{b^2 - 4ac}{4a^2}} \\
x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}
\]

THE QUADRATIC FORMULA

Given any quadratic equation in the standard form \( ax^2 + bx + c = 0 \), where \( a \neq 0 \), the solutions are given by the following formula:

\[
x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}
\]

You’ve seen that a quadratic equation can have two, one, or no real solutions. In the quadratic formula, the expression under the radical, \( b^2 - 4ac \), determines the number and nature of the solutions.

DEFINITION

For a quadratic equation in the standard form \( ax^2 + bx + c = 0 \), the **discriminant** is **\( b^2 - 4ac \)**.

- **positive** \( (b^2 - 4ac > 0) \) **two real roots**
- **zero** \( (b^2 - 4ac = 0) \) **one real root**
- **negative** \( (b^2 - 4ac < 0) \) **no real roots, but two complex roots**
Example 5  Use the quadratic formula to solve.

A  $3x^2 - 12x + 6 = 0$

**Solution** Identify the coefficients and substitute into the quadratic identify formula.

$a = 3, \ b = -12, \ c = 6$

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \cdot 3 \cdot 6}}{2 \cdot 3} \]

\[ x = \frac{12 \pm \sqrt{72}}{6} \]

\[ x = \frac{12 \pm 6\sqrt{2}}{6} \]

\[ x = 2 \pm \sqrt{2} \]

The solution set is \( \{2 + \sqrt{2}, 2 - \sqrt{2}\} \)

B  $2x^2 + 7 = -5x$

**Solution** First write the equation in standard form: $2x^2 + 5x + 7 = 0$. Then identify the coefficients and substitute into the formula.

\[ a = 2, \ b = 5, \ c = 7 \]

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ x = \frac{-5 \pm \sqrt{5^2 - 4 \cdot 2 \cdot 7}}{2 \cdot 2} \]

\[ x = \frac{-5 \pm \sqrt{25 - 56}}{4} \]

\[ x = \frac{-5 \pm i\sqrt{31}}{4} \]

The solution set is \( \left\{ \frac{-5 + i\sqrt{31}}{4}, \frac{-5 - i\sqrt{31}}{4} \right\} \)

The approximate solutions in Example 5A above are 3.41 and 0.59.

Problem Set

Use the graph to estimate the solutions of \( f(x) = 0 \).
Substitute to check that your estimates are reasonable.

1. \[ f(x) = 2x^2 + 4x - 1 \]
2. \[ f(x) = x^2 - 2.5x + 3 \]
Use factoring to solve.
5. \(-6x + 3 = -3x^2\)
6. \(x^2 + 12x + 20 = -16\)
7. \(2x^2 - 5x - 3 = 0\)

Complete the square to solve.
11. \(x^2 - x - 6 = 0\)
12. \(4x^2 + 20x + 9 = 0\)
13. \(2x^2 - 16x + 44 = 0\)

Use the quadratic formula to solve.
17. \(2x^2 - 7x + 5 = 0\)
18. \(3x^2 + 1 = -2x\)
19. \(x^2 = 5 + 4x\)

Use the discriminant to classify the number and types of solutions of each equation.
23. \(3x^2 - x + 5 = 0\)
24. \(x^2 = 8x - 2\)
25. \(6x + 3 = -3x^2\)
26. \(6x^2 = 7x - 2\)

*27. Challenge \((\sqrt{5}x - 1)(\sqrt{5}x + 3) = 0\)
*28. Challenge \((\sqrt{7}x + 2)^2 = 0\)
Quadratic Inequalities

A quadratic inequality involves a quadratic expression and an inequality symbol (<, >, ≤, ≥, or ≠).

Quadratic Inequalities in One Variable

Any value of the variable that makes an inequality true is a solution for the inequality. You can describe the solution set of an inequality in one variable by using set notation or by graphing it on a number line.

Example 1  Determine which of \( x = -4, \) \( x = 0, \) and \( x = 3 \) are solutions to \( x^2 + 2x - 3 < 0. \)

Solution  Substitute \( x = -4, \) \( x = 0, \) and \( x = 3 \) into the inequality \( x^2 + 2x - 3 < 0: \)

\[
(-4)^2 + 2(-4) - 3 < 0 \quad 0^2 + 2 \cdot 0 - 3 < 0 \quad 3^2 + 2 \cdot 3 - 3 < 0
\]

\[
5 \checkmark \quad -3 < 0 \checkmark \quad 12 \checkmark 0
\]

Only \( x = 0 \) is a solution, since \( x = -4 \) and \( x = 3 \) do not satisfy the inequality.

\[\text{TIP}\]

There are infinitely many solutions to the inequality in Example 1. The value 0 is only one of them.

**HOW TO GRAPH THE SOLUTION TO AN INEQUALITY IN ONE VARIABLE**

**Step 1** Find the real solutions of the related equation. Plot them as open dots for a strict inequality or closed dots for a nonstrict inequality.

**Step 2** These points divide the number line into intervals. Choose one test point from each interval.

**Step 3** Either all the points in an interval are solutions or none of them is. Test each point; if it is a solution, shade the corresponding interval.

Example 2  Graph \( 2x^2 - 5x \geq -2. \) Write the solution set using set notation.

Solution  The related equation, in standard form, is \( 2x^2 - 5x + 2 = 0. \) Its solutions will be the boundary points of the solution intervals:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
= \frac{-( -5) \pm \sqrt{(-5)^2 - 4 \cdot 2 \cdot 2}}{2 \cdot 2}
\]

\[
= \frac{5 \pm \sqrt{9}}{4} = 2, 0.5
\]

Plot 2 and 0.5 with closed dots.

REMEMBER

Sometimes you can solve the related equation using factoring. In Example 2, however, you need to use the quadratic formula.
Step 2 Choose a value to represent each of the three resulting intervals. Substitute each value of \( x \) into \( 2x^2 - 5x \geq -2 \):

\[
2 \cdot 0^2 - 5 \cdot 0 \geq -2 \quad 2 \cdot 1^2 - 5 \cdot 1 \geq -2 \quad 2 \cdot 3^2 - 5 \cdot 3 \geq -2
\]

\[
0 \geq -2 \quad -3 \not\geq -2 \quad 3 \geq -2
\]

Step 3 Shade the interval containing \( x = 0 \). Do not shade the interval containing \( x = 1 \). Shade the interval containing \( x = 3 \).

The solution set of \( 2x^2 - 5x \geq -2 \) is \( \{ x \mid x \leq 0.5 \text{ or } x \geq 2 \} \). ■

Quadratic Inequalities in Two Variables

Some quadratic inequalities involve two variables. The solutions to these inequalities are usually graphed as regions in the Cartesian plane.

Example 3 Determine whether \((-2, 0)\) and \((5, 12)\) are solutions of \( y < x^2 - 4x + 7 \).

Solution Substitute each pair of \( x \)- and \( y \)-coordinates into \( y < x^2 - 4x + 7 \).

Test \((-2, 0)\): \( 0 < (-2)^2 - 4(-2) + 7 \)

\[
0 < 4 + 8 + 7 \quad 0 < 19
\]

This is true, so the point \((-2, 0)\) is a solution of \( y < x^2 - 4x + 7 \).

Test \((5, 12)\): \( 12 < 5^2 - 4 \cdot 5 + 7 \)

\[
12 < 25 - 20 + 7 \quad 12 < 12
\]

Twelve is not less than itself. So \((5, 12)\) is not a solution. ■

HOW TO GRAPH THE SOLUTION TO AN INEQUALITY IN TWO VARIABLES

Step 1 Draw the related curve. Use a dashed curve for a strict inequality \(<, >\); use a solid curve for a nonstrict inequality \(\leq, \geq\).

Step 2 Choose a test point in each region of the plane created by the curve.

Step 3 Either all the points in a region are solutions or none of them is. Test each point; if it is a solution, shade the corresponding region.

QUADRATIC INEQUALITIES
REMEMBER

There are a number of ways to graph this parabola.
1. You can find the vertex by completing the square:
   \[ y = -(x + 2)^2 + 9. \]
2. You can also factor to find the roots, \( x = -5 \) and 1:
   \[ y = -(x + 5)(x - 1). \]
The average of the roots is the axis of symmetry, \( x = -2 \); use this value to find the vertex \((-2, 9)\).

Example 4  Graph each inequality.

A  \[ y > -x^2 - 4x + 5 \]

Solution

Step 1  The graph of the equality \( y = -x^2 - 4x + 5 \) is a parabola. Use any strategy to graph it, drawing a dashed curve for the strict inequality >.

Step 2  Choose and test a point above the parabola, such as \((2, 3)\), and a point below the parabola, such as \((0, 0)\):

Try \((2, 3)\):
\[
y > -x^2 - 4x + 5
\]
\[
3 \gtrsim -2^2 - 4 \cdot 2 + 5
\]
\[
3 > -7 \checkmark
\]
Try \((0, 0)\):
\[
y > -x^2 - 4x + 5
\]
\[
0 \gtrsim -0^2 - 4 \cdot 0 + 5
\]
\[
0 \not> 5
\]

Step 3  \((2, 3)\) is a solution. Shade this region. \((0, 0)\) is not a solution. Do not shade this region. ■

B  \[ y \geq \frac{1}{2}(x - 6)^2 + 3 \]

Solution

Step 1  The graph of \( y = \frac{1}{2}(x - 6)^2 + 3 \) is a parabola with its vertex at \((6, 3)\). Draw a solid curve, since points on the curve are solutions to the nonstrict inequality ≥.

Step 2  Test points inside and outside the parabola.

Try \((6, 5)\):
\[
y \geq \frac{1}{2}(x - 6)^2 + 3
\]
\[
5 \gtrsim \frac{1}{2}(6 - 6)^2 + 3
\]
\[
5 \gtrsim 3 \checkmark
\]
Try \((6, 2)\):
\[
y \geq \frac{1}{2}(x - 6)^2 + 3
\]
\[
2 \gtrsim \frac{1}{2}(6 - 6)^2 + 3
\]
\[
2 \not\geq 3
\]

Step 3  \((6, 5)\) is a solution. Shade inside. \((6, 2)\) is not a solution. Do not shade outside. ■
Graphing a Quadratic Equation to Solve an Inequality in One Variable

Example 5  Solve the inequality \( x^2 + 2x - 3 < 0 \).

Solution  The inequality \( x^2 + 2x - 3 < 0 \) corresponds to \( x \)-values that make \( y < 0 \) for the function \( y = x^2 + 2x - 3 \). Find the \( x \)-intercepts of this function, where \( y = 0 \):

\[
\begin{align*}
x^2 + 2x - 3 &= 0 \\
(x + 3)(x - 1) &= 0 \\
x &= -3 \text{ or } x = 1
\end{align*}
\]

The solution set of \( x^2 + 2x - 3 < 0 \) is the part of the graph for which \( y < 0 \), shown in red below the \( y \)-axis. These values are \( \{x \mid -3 < x < 1\} \), also shown on the number line.

Using a Quadratic Inequality to Solve a Real-World Problem

Quadratic inequalities have many applications.

Example 6  Paper is made of pulp. The strength, \( y \), of brown wrapping paper in pounds per square inch (psi) is related to the percentage of hardwood, \( x \), in the pulp by the function \( y = -0.63x^2 + 11.76x - 6.67 \). What percentages of hardwood result in paper with a strength of at least 40 psi?

Solution  For a strength of at least 40 psi, \( y \geq 40 \). So you need to solve the inequality \( -0.63x^2 + 11.76x - 6.67 \geq 40 \).

Method 1  You can solve this inequality by testing intervals on a number line. First write the inequality in standard form by subtracting 40 from both sides: \( -0.63x^2 + 11.76x - 46.67 \geq 0 \). Then find the boundary points by solving the related equation: \( -0.63x^2 + 11.76x - 46.67 = 0 \).

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
= \frac{-11.76 \pm \sqrt{11.76^2 - 4 \cdot (-0.63) \cdot (-46.67)}}{2 \cdot (-0.63)}
\]

\[
\approx 5.72 \text{ or } 12.94
\]
Plot these values with closed dots, and test values in each interval.

Test 4:  
\[ -0.63x^2 + 11.76x - 6.67 \geq 40 \]
Values: \( x = -2, 0, 3 \)

Test 10:  
\[ -0.63x^2 + 11.76x - 6.67 \geq 40 \]
Values: \( x = -3, 0, 2 \)

Test 14:  
\[ -0.63x^2 + 11.76x - 6.67 \geq 40 \]
Values: \( x = -1, 0, 4 \)

\[ 30.29 \not\geq 40 \]
\[ 47.93 \geq 40 \checkmark \]
\[ 34.49 \not\geq 40 \]

Method 2 Graph \( y = -0.63x^2 + 11.76x - 46.67 \) with a calculator or software and find the points of intersection. The solutions to the inequality are the points where \( y \) intersects or is above the \( x \)-axis.

Both methods show that the solution set is approximately \( \{ x \mid 5.72 \leq x \leq 12.94 \} \). The paper’s strength is at least 40 psi for pulp that is between 5.72% and 12.94% hardwood.

### Problem Set

Determine whether the given values or points are solutions to the given quadratic inequality.

1. Inequality: \( x^2 - 5 > 0 \)
   Values: \( x = -2, 0, 3 \)

2. Inequality: \( 2x^2 - x - 1 < 0 \)
   Values: \( x = -3, 0, 2 \)

3. Inequality: \( -x^2 + 4x - 3 < 0 \)
   Values: \( x = -1, 0, 4 \)

4. Inequality: \( \frac{1}{3}x^2 - 2x + 1 > 0 \)
   Values: \( x = -1, 0, 3 \)

5. Inequality: \( x^2 - 3x + 1 > y \)
   Points: \((5, 0), (-2, 4)\)

6. Inequality: \( y \leq x^2 - 2x + 7.5 \)
   Points: \((1, 10), (4, 7)\)

7. Inequality: \( 2y < x^2 + 10x + 5 \)
   Points: \((-8.5, -2), (-3, 2)\)

8. Inequality: \( 3x^2 - 7x - 9 \leq y \)
   Points: \((-3, 3), (-1, 4)\)

9. Inequality: \( y < \frac{1}{2}x^2 + 4x - 3 \)
   Points: \((-2, 2), (0, -5)\)

Solve the quadratic inequality. Graph the solution on a number line and write the solution in set notation.

10. \( 2x^2 - 4x > 0 \)

11. \( -x^2 - 3x + 4 \leq 0 \)

12. \( 2x^2 - 16x + 32 > 0 \)

13. \( 3x^2 - 4 \leq x \)

14. \( 4x + 1 - 2x^2 \)

15. \( x^2 - 5 < 4x \)

16. \( x^2 - 4 < 0 \)

17. \( 2x^2 + x - 1 < 0 \)

18. \( x^2 \geq 3x + 10 \)
For each quadratic inequality in two variables, graph the solution set on the Cartesian plane.

19. \( y \leq x^2 - 3x + 2 \)
20. \( y < x^2 + 5x + 4 \)
21. \( y - x^2 > -12x + 32 \)
22. \( y \geq \frac{1}{2}(x - 4)^2 + 2 \)
23. \( y < 2(x - 1)(x - 2) \)
24. \( 2y \leq -2x^2 - 7x + 4 \)
25. \( y \geq -x^2 - x + 20 \)
26. \( y \geq 3x^2 - \frac{21}{2}x + \frac{9}{2} \)
27. \( y \leq -2x^2 + 14x - 20 \)

Solve.

28. The population, in thousands, of Bronson’s town Maple Woods can be modeled by the function \( y = 0.45x^2 - 3x + 16 \), where \( x \) is Bronson’s age. For what ages was the population of his town 12,000 or fewer?

29. A ball is launched vertically upward. The height, \( y \), of the ball in meters is related to the amount of time, \( x \), the ball is in the air in seconds by the function \( y = -9.8x^2 + 30x + 5 \). For which times does the ball reach a height of at least 25 meters?

Write a quadratic inequality or system of inequalities that corresponds to the given graph.

30. Kayaking Supplies’ unit cost, \( y \), of producing kayaks is related to the number of kayaks made, \( x \), by the function \( y = 0.08x^2 - 5.2x + 136 \). What numbers of kayaks will result in a unit cost of more than $100?

31. Wind exerts pressure on the sides of buildings. The pressure, \( y \), of the wind in pounds of force per square foot is related to the wind speed, \( x \), in miles per hour by the function \( y = 0.003x^2 - 0.0003x + 1.71 \). What wind speeds will result in a pressure of less than 10 pounds of force per square foot?
Finding a Quadratic Function from Points

It takes any two points to determine a line, but two points is not usually enough to determine a quadratic function. There is one and only one quadratic function graph that contains three given points, as long as those points all have different first coordinates and do not all lie on a line.

Finding a Quadratic Function from Two $x$-intercepts and a Point

To find a quadratic equation from two $x$-intercepts and one point, substitute the values of the $x$-intercepts into the factored form of the equation. Then use the additional point to solve for $a$.

**Example 1**  Find the equation of the quadratic function that contains the given $x$-intercepts and point.

$x$-intercepts: $-5$ and $1$, point: $(2, 7)$

**Solution**  The function contains $(-5, 0)$ and $(1, 0)$ because the $x$-intercepts are $-5$ and $1$.

\[
y = a(x - r_1)(x - r_2) \quad \text{Quadratic equation in factored form}
\]

\[
y = a(x - (-5))(x - 1) \quad \text{Substitute } -5 \text{ and } 1 \text{ for } r_1 \text{ and } r_2.
\]

\[
y = a(x + 5)(x - 1) \quad \text{Now only the coefficient } a \text{ is unknown.}
\]

\[
7 = a(2 + 5)(2 - 1) \quad \text{Substitute the point } (2, 7) \text{ for } x \text{ and } y \text{ to find } a.
\]

\[
7 = a \cdot 7 \cdot 1 \quad \text{Simplify.}
\]

\[
a = 1 \quad \text{Solve for } a.
\]

Substitute $a = 1$ to get the function in factored form: $y = (x + 5)(x - 1)$.

You could also expand this to get standard form: $y = x^2 + 4x - 5$. ■
x-intercepts: $-6$ and $\frac{1}{2}$, point: $(2, 8)$

**Solution**  The function contains $(-6, 0)$ and $\left(\frac{1}{2}, 0\right)$ because the x-intercepts are $-6$ and $\frac{1}{2}$.

$y = a(x - r_1)(x - r_2)$  Quadratic equation in factored form

$y = a(x - (-6))(x - \frac{1}{2})$  Substitute $-6$ and $\frac{1}{2}$ for $r_1$ and $r_2$ in the factored form.

$y = a(x + 6)\left(x - \frac{1}{2}\right)$  Now only the coefficient $a$ is unknown.

$8 = a(2 + 6)\left(2 - \frac{1}{2}\right)$  Substitute the point $(2, 8)$ for $x$ and $y$ to find $a$.

$8 = a \cdot 8 \cdot \frac{3}{2}$

$a = \frac{2}{3}$

Substitute $a = \frac{2}{3}$ to get the function in factored form: $y = \frac{2}{3}(x + 6)\left(x - \frac{1}{2}\right)$.

You could also expand this to get standard form: $y = \frac{2}{3}x^2 + \frac{11}{3}x - 2$. ■

**Finding a Quadratic Function from a Vertex and a Point**

To find a quadratic equation when given a vertex and one other point, substitute the ordered pair of the vertex into the vertex form of the equation. Then use the additional point to solve for $a$.

**Example 2**  Find the equation of the quadratic function that has vertex $(-2, 5)$ and contains the point $(-3, 1)$.

**Solution**  Substitute the vertex $(-2, 5)$ for $h$ and $k$ in the vertex form.

$y = a(x - (-2))^2 + 5$  Substitute $(-3, 1)$ for $x$ and $y$ to find $a$.

$1 = a(-3 + 2)^2 + 5$

$1 = a(-1)^2 + 5$

$1 = a \cdot 1 + 5$

$-4 = a$

Substitute $a$ to get the function in vertex form: $y = -4(x + 2)^2 + 5$.

You could also expand to get standard form: $y = -4x^2 - 16 - 11$. ■

**THINK ABOUT IT**

In Examples 1 and 2, verify that the quadratic equation contains the given points.
Finding a Quadratic Function from Three Points

You can find a quadratic function from three points by using a system.

**Example 3** Find the equation of the quadratic function that contains \((-5, 30), (-2, -3),\) and \((1, 0)\).

**Solution** Substitute the given points for \(x\) and \(y\) into the standard form \(y = ax^2 + bx + c\) to form a system of three linear equations in \(a, b,\) and \(c\).

<table>
<thead>
<tr>
<th>Point</th>
<th>Substitution</th>
<th>Linear equation in (a, b,) and (c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-5, 30))</td>
<td>(30 = a(-5)^2 + b(-5) + c)</td>
<td>(30 = 25a - 5b + c) (I)</td>
</tr>
<tr>
<td>((-2, -3))</td>
<td>(-3 = a(-2)^2 + b(-2) + c)</td>
<td>(-3 = 4a - 2b + c) (II)</td>
</tr>
<tr>
<td>((1, 0))</td>
<td>(0 = a \cdot 1^2 + b \cdot 1 + c)</td>
<td>(0 = a + b + c) (III)</td>
</tr>
</tbody>
</table>

In the equation \(y = ax^2 + bx + c\), the letters \(a, b,\) and \(c\) are coefficients, but in Equations I, II, and III, they are variables. Eliminate a variable in the 3 by 3 system to obtain a 2 by 2 system.

**Step 1** Multiply Equation II by \(-1\), and then add the resulting equation and Equation I. This eliminates \(c\).

\[
\begin{align*}
30 &= 25a - 5b + c \quad \text{(I)} \\
+3 &= -4a + 2b - c \quad \text{I} \cdot \text{II} \\
33 &= 21a - 3b \quad \text{(IV)}
\end{align*}
\]

**Step 2** Multiply Equation II by \(-1\), and then add the resulting equation and Equation III. This also eliminates \(c\).

\[
\begin{align*}
3 &= -4a + 2b - c \quad \text{I} \cdot \text{II} \\
+0 &= a + b + c \quad \text{(III)} \\
3 &= -3a + 3b \quad \text{(V)}
\end{align*}
\]

Equations IV and V form a 2 by 2 system. Solve the system.

**Step 3** Add Equations IV and V to eliminate \(b\). Then solve for \(a\).

\[
\begin{align*}
33 &= 21a - 3b \quad \text{(IV)} \\
+3 &= -3a + 3b \\
36 &= 18a \\
2 &= a
\end{align*}
\]

**Step 4** Substitute 2 for \(a\) in either Equation IV or Equation V. Solve for \(b\).

\[
\begin{align*}
3 &= -3a + 3b \quad \text{(V)} \\
3 &= -3 \cdot 2 + 3b \\
3 &= b
\end{align*}
\]

**Step 5** Substitute 2 for \(a\) and 3 for \(b\) in any of the original three equations. Solve for \(c\).

\[
\begin{align*}
0 &= a + b + c \quad \text{(III)} \\
0 &= 2 + 3 + c \\
-5 &= c
\end{align*}
\]

**Step 6** Substitute \(a, b,\) and \(c\) in the standard form to get the quadratic function:

\[y = ax^2 + bx + c \quad \Rightarrow \quad y = 2x^2 + 3x - 5\]
**Problem Set**

Find the equation of the quadratic function that has the given $x$-intercepts and point.

1. $x$-intercepts: $-1$ and $3$
   point: $(2, -6)$
2. $x$-intercepts: $-2$ and $4$
   point: $(3, -15)$
3. $x$-intercepts: $-3$ and $1$
   point: $(-1, -4)$
4. $x$-intercepts: $-2$ and $2$
   point: $(1, 6)$
5. $x$-intercepts: $-1$ and $4$
   point: $(3, -4)$
6. $x$-intercepts: $1$ and $3$
   point: $(4, 6)$
7. $x$-intercepts: $2$ and $4$
   point: $(1, -3)$
8. $x$-intercepts: $-9$ and $-2$
   point: $(-1, 3)$
9. $x$-intercepts: $\frac{3}{2}$ and $4$
   point: $\left(5, \frac{7}{4}\right)$

Find the equation of the quadratic function that has the given vertex and point.

10. vertex: $(1, -1)$
    point: $(3, 3)$
11. vertex: $(-1, -2)$
    point: $(1, 2)$
12. vertex: $(-2, 3)$
    point: $(1, 21)$
13. vertex: $(2, -1)$
    point: $(3, 1)$
14. vertex: $(3, 1)$
    point: $(5, -3)$
15. vertex: $(-2, -1)$
    point: $(2, -17)$
16. vertex: $(0, 0)$
    point: $(1, 3)$
17. vertex: $(1, -1)$
    point: $(0, -4)$
18. vertex: $(-1, 2)$
    point: $(2, -25)$

Find the equation of the quadratic function that contains the given three points.

19. $(0, 2), (-1, 5), (2, 2)$
20. $(0, -1), (1, 1), (2, 1)$
21. $(-1, 8), (1, 2), (2, 2)$
22. $(-2, 4), (0, 2), (1, -2)$
23. $(-1, -4), (1, 0), (2, 5)$
24. $(-1, 1), (0, 2), (1, 7)$
25. $(0, -3), (1, -2), (2, 5)$
26. $(-1, -1), (0, 2), (1, 1)$
27. $(-1, 3), (0, -1), (1, 1)$

Solve.

**28. Challenge** Find the cubic function that contains points $(-1, 0), (0, 1), (1, 0)$, and $(2, 3)$.

**29. Challenge** Develop a general rule explaining how many points are needed to define a polynomial with degree $n$. 
Applications: Quadratic Functions

Quadratic functions can be used to model many real-world situations.

For some problems, you can create and solve a quadratic equation; for other problems, you can create and optimize a quadratic function.

Application: Furniture Dimensions

Example 1  Television screens are described by their diagonal measurement and by their length-to-width ratio. Many widescreen televisions have a length-to-width ratio (usually called the aspect ratio) of 16 : 9. If Mr. Scott has a 40-inch widescreen television with an aspect ratio of 16 : 9, could he fit it in a piece of furniture with an opening that is 37 inches by 24 inches?

Solution  Draw a rectangular television screen. The 40-inch diagonal divides it into two congruent right triangles. Since the ratio of the sides is 16 : 9, the sides can be represented by the multiples 16x and 9x.

Use the Pythagorean theorem to set up a quadratic equation. Simplify the equation and solve for x.

\[ a^2 + b^2 = c^2 \]
\[ (9x)^2 + (16x)^2 = 40^2 \]
\[ 81x^2 + 256x^2 = 1600 \]
\[ 337x^2 = 1600 \]
\[ x = \pm \sqrt{\frac{1600}{337}} \approx \pm 2.18 \]

Because negative length is not helpful in this situation, \(-2.18\) is an extraneous solution. So the television’s dimensions are

\[ 16x \approx 16 \cdot 2.18 = 34.88 \]
\[ 9x \approx 9 \cdot 2.18 = 19.62 \]

The dimensions of the 40-inch widescreen television are approximately 35 inches by 20 inches. Mr. Scott’s furniture can accommodate a 40-inch widescreen as long as the frame around the screen is no more than 1 inch wide on each side. ■
Application: Projectile Motion

HEIGHT OF PROJECTILE

The height $h$ of a projectile after $t$ seconds is given by the function

$$h(t) = \frac{1}{2}gt^2 + v_0 t + h_0,$$

where $v_0$ is the initial vertical velocity, $h_0$ is the initial height, and $g$ is the downward acceleration due to gravity (on the earth, $g \approx -32 \text{ ft/s}^2 \approx -9.8 \text{ m/s}^2$).

Example 2  Kyle throws a baseball straight up from a height of 4 feet with an initial speed of 30 feet per second.

A Write an equation to model the height $y$ of the ball after $t$ seconds.

B Make a graph of the baseball’s height as a function of time.

C What is the maximum height of the baseball?

D If Kyle doesn’t catch the ball, when will the ball hit the ground?

Solution

A Use the values $g = -32$, $v_0 = 30$, and $h_0 = 4$ to write a function for the height of Kyle’s baseball:

$$h(t) = -\frac{1}{2} \cdot 32t^2 + 30t + 4 = -16t^2 + 30t + 4$$

B Make a graph using any tool, such as a table of values, software, or a calculator.

C The maximum height (the maximum value of the function) corresponds to the $h$-value of the vertex. You can find the vertex by using your knowledge of the properties of quadratic functions:

$$t = -\frac{b}{2a} = -\frac{30}{2 \cdot (-16)} = \frac{15}{16} = 0.9375$$

The $h$ coordinate of the vertex is $h(0.9375) \approx 18.1$.

Therefore, the ball reaches a maximum height of about 18.1 feet.

D The ground is height $h = 0$, so the ball is on the ground when $0 = -16t^2 + 30t + 4$.

Use the graph as a guide. The graph shows two roots. The leftmost root is extraneous, since negative time has no meaning in this problem. The positive root seems to be at $t = 2$, so the positive root seems to be at $t = 2$.

Check to see if $t = 2$ yields a height of 0:

$$h(2) = -16 \cdot 2^2 + 30 \cdot 2 + 4$$

$$= -64 + 60 + 4$$

$$= 0$$

So the ball hits the ground 2 seconds after Kyle throws it. ■

THINK ABOUT IT

Keep in mind that the horizontal axis represents time, not horizontal distance. In fact, in this graph, Kyle’s ball only moves up and down, not forward.

THINK ABOUT IT

You could also use factoring or the quadratic formula to solve this problem.
Application: Optimizing Area

Example 3  Selena bought 12 meters of fencing to make a rectangular pen for her rabbit. She will use the wall of her shed as one side of the pen and the fencing for the other three sides. What dimensions maximize the pen’s area?

Solution  Sketch and label a diagram of the pen. Let $x$ represent the length of the two sides perpendicular to the shed. Of the 12 meters of fencing, $12 - 2x$ remain for the third side. So the dimensions of the pen are $x$ and $12 - 2x$.

Use the area formula for a rectangle to set up a quadratic function, and simplify.

\[
A = x \cdot (12 - 2x) \\
A = -2x^2 + 12x
\]

Because the leading coefficient is negative, the graph of $A$ would be a parabola that opens down; the coordinates of the vertex give the dimensions that yield the maximum area. The $x$-value of the vertex is

\[
x = -\frac{b}{2a} = -\frac{12}{2 \cdot (-2)} = 3
\]

Therefore, the pen will have maximum area when its dimensions are $x = 3$ meters and $12 - 2x = 6$ meters. This gives an area of 18 square meters. ■

Problem Set

Solve each problem by using a quadratic equation.

1. Jo can grow 5 pounds of wheat per square meter of farmland. If Jo grows 2000 pounds of wheat on a square farm, what is the length of the farm?

2. An open box is formed by cutting squares of length 4 inches out of a square piece of paper. The resulting box has a volume of 400 cubic inches. What are the dimensions of the original piece of paper?

3. Suppose a 25-foot ladder is leaning against a wall such that it reaches the wall 17 feet farther than the distance from the base of the ladder to the wall. What is the distance from the base of the ladder to the wall?

4. A rectangular warehouse covers 20,000 square feet of ground. The warehouse is 100 feet wider than its depth. What is the width of the warehouse?

5. Suppose a sheep requires 2 square meters of space to graze. Jose has 25 sheep and decides to make his sheep pen 5 meters longer than it is wide. What is the width of the sheep pen?

6. Sarah and Nakia are racing to their homes. Sarah’s home is due north and Nakia’s home is due east. Nakia gives Sarah a 7-meter head start. Assuming they run at the same rate, how far will Nakia run before the distance between the two girls is 17 meters?

7. A rectangular park has dimensions of 50 km by 120 km. The park has a rectangular grass field surrounded by woods. What are the dimensions of the field if the area of the field is a quarter of the total area of the park? Assume that the width-length ratio in the field is equal to the width-length ratio in the park.
For each projectile motion problem, do the following:
A. Write a function equation to model the problem.
B. Graph the function.
C. Answer the questions.

8. Silas tosses a football up at 10 feet per second, from an initial height of 6 feet. What is the maximum height? If no one catches the ball, when will the ball hit the ground?

9. Marques drops a plate from a height of 5 feet. When will the plate hit the ground?

10. Krystal throws a rappelling rope down a 50-meter cliff at 10 meters per second. When will the rope hit the ground?

11. Santino fires a paintball 3 meters from the ground such that the initial trajectory is perfectly horizontal. Assume the paintball does not hit anything and the ground is level. When will the paintball hit the ground?

12. Khalid jumps out of an airplane at a height of 10,000 feet. If Khalid has to release his parachute at 5000 feet, at what time should he do this?

13. David throws a bucket with an initial speed of 2 meters per second into a well that is 30 meters deep. When will the bucket hit the bottom of the well?

14. Nikolai kicks a soccer ball straight up at a speed of 20 meters per second. What is the maximum height? When will the ball hit the ground?

Solve each optimization problem.

15. Brody’s farm borders a river. He buys 1000 meters of fencing. What dimensions maximize the area of the farm?

16. A rectangular house is designed with two parallel interior walls. The contractor has enough material to build 800 meters of walls. What is the maximum area of the house?

17. Flags & Designs, Inc., has enough brick to build 400 meters of exterior walls for its new store. What is the maximum area of the store that the company can build?

18. Suppose the daily cost of farming \( x \) pounds of potatoes is approximated by \( C(x) = 0.2x^2 - 5x + 200 \). How many pounds of potatoes would minimize the costs of producing the potatoes?

19. Aleah’s garden borders a wall. She buys 80 meters of fencing. What dimensions maximize the area of the garden?

20. Suppose the daily cost of producing \( x \) computers is approximated by \( C(x) = 0.4x^2 - 20x + 210 \). How many computers would minimize the production costs?

21. Felix sells dishwashers for $400 per machine. He averages about 12 sales a day. If Felix loses a customer for every $20 he increases the price, what is the price that maximizes his profits?

22. **Challenge** An open box is created by cutting out squares from a piece of paper that is 10 inches by 10 inches. What are the dimensions of the cutouts that maximize the volume of the box?

23. **Challenge** A cylinder with no top has a surface area of \( 6\pi \) square meters. What radius produces the largest possible volume of the cylinder?

24. **Challenge** Suppose a missile is launched from the ground (or \( (0, 0) \)) and follows the path \( y = 20x \) initially. At what point on the coordinate grid will the missile be closest to the point \( (30, 10) \)?